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Compact and noncompact dynamics of the $SU(1,1)$ coherent states driven by a coherence preserving Hamiltonian

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Abstract

We investigate dynamics of the $SU(1,1)$ coherent states with the use of the group transformations diagonalizing the coherence preserving Hamiltonian driving the physical system. The model physical system we consider may be viewed as a particular case of the generalized time-dependent harmonic oscillator, or as a generalization of the degenerate parametric amplifier, with the pumping field having modulated amplitude and a nonresonant phase. A Hamiltonian of such a system is given as a linear combination of the $SU(1,1)$ generators with time-dependent coefficients, and the group transformations, mentioned above, transform this Hamiltonian to an expression containing only one generator, i.e. diagonalize the Hamiltonian. Trajectories of the complex coherent state parameter in the phase space (Lobachevskii plane) can be divided into two classes: compact trajectories never approaching the unit circle (boundary of the phase space) and noncompact trajectories approaching the unit circle from inside asymptotically, after sufficiently long time. The character of the dynamics is reflected by the time behaviour of the parameters of the group transformation diagonalizing the Hamiltonian. The main observation is that in the case of noncompact dynamics absolute values of the group parameters increase indefinitely, although in some cases they may exhibit a singular behaviour with regions of rapid variations, sudden changes of their values or cusplike singularities. For compact dynamics the group transformation parameters remain bounded and exhibit an oscillatory behaviour as functions of time.

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1. Introduction

The purpose of this paper is to study dynamics of the coherent states of a model physical system with a Hamiltonian given as a linear combination of the $SU(1,1)$ group generators with time-dependent coefficients. Examples of physical systems which can be described by a Hamiltonian of this type may be given by the time-dependent harmonic oscillator or degenerate parametric amplifier. The dynamics of such physical systems in general, and their coherent states in particular, has been extensively studied from various points of view in the past. A general analysis of the dynamical systems with time-dependent Hamiltonians has been worked out by Lewis and Riesenfeld [1] with the use of time-dependent invariants having time-independent eigenvalues. In the same paper this method was used to analyse, among other things, the time-dependent harmonic oscillator. The quadratic Hamiltonians were also studied, among other things, by Xiao-Chun Gao *et al* [2], where the exact solution for the generalized time-dependent harmonic oscillator was found, de Toledo-Pisa [3], who studied time evolution of mean values of the observables or, recently, Cervero and Lejaretta [4], who discussed the canonical formalism for the time-dependent harmonic oscillator. The adiabatic phase in physical systems described by quadratic Hamiltonians was analysed by Jackiw [5], Xiao-Chun Gao [2] and de Sousa Gerbert [6].

A general Hamiltonian bilinear in coordinates and momenta (or, equivalently, quadratic in the annihilation and creation operators of the harmonic oscillator) can be expressed as a linear combination of the $SU(1,1)$ group generators. Therefore, the group-theoretical methods using this particular dynamical group are of great usefulness in analysing dynamics of such systems. For a review of the applications of algebraic methods to the description of various dynamical systems, among them the time-dependent harmonic oscillator, see for instance [7]. Time evolution of the harmonic oscillator with time-dependent frequency was also studied recently by means of group theoretical methods by Penna [8], where unitary $SU(1,1)$-group transformations diagonalizing the Hamiltonian were used. In the present paper we analyse dynamics of the $SU(1,1)$ coherent states using in fact methods similar to those developed in [8].

A systematic classification of the coherent states can be based on group theory [9]. For instance, coherent states related to the classical states of a quantum harmonic oscillator can be constructed from the Weyl–Heisenberg group H_4. A general method for constructing coherent states for an arbitrary Lie group has been given by Perelomov [10], where the coherent states were defined as generated by operating on the vacuum state with an appropriately chosen displacement operator. Of particular importance are the coherent states of the $SU(1,1)$ group, which can be characterized by diminished quantum fluctuations of one of the canonically conjugate variables (squeezed states).

Classical dynamics of the $SU(1,1)$ coherent states, generated by a classical Hamiltonian given as the coherent state expectation value of the quantum Hamiltonian, has been also studied recently in various contexts. In particular, the Hamiltonian linear in the group generators [11] is a coherence preserving Hamiltonian [12], i.e. an initially $SU(1,1)$-coherent state remains coherent during time evolution. To give a few examples of the classical dynamics of $SU(1,1)$ Hamiltonians, Gerry and Silverman [13] and Gerry [14] gave the path-integral formulation of the dynamics of $SU(1,1)$ coherent states, and classical dynamics of the $SU(1,1)$ Hamiltonians linear and quadratic in the group generators was investigated by Gerry and Kiefer [15, 16] and Gortel and Turski [17]. Application of the classical dynamics of $SU(1,1)$ coherent states to the degenerate parametric amplifier can be found, for instance, in [12].

Presently we shall study *phase space* dynamics of $SU(1,1)$ coherent states using a method of unitary group transformations. A similar method has been used by Penna [8] in the investigation of compact and noncompact dynamics of the time-dependent harmonic oscillator. The Hamiltonian we use here, given as a linear combination of $SU(1,1)$ generators, is similar to that studied by Dattoli *et al* [18] with the time-dependent coefficients which will be specified later. It can, in fact, be viewed as a generalization of the degenerate amplifier Hamiltonian with the pumping field amplitude dependent on time, and with nonresonant frequency, not equal to the frequency of the quantized field mode. In this case no analytic solutions of the evolution equations are known and numerical methods have to be used. The phase space of the $SU(1,1)$ coherent states is a curved space, in fact the Lobachevskii plane, represented by the interior of the unit circle with non-Euclidean geometry [13, 19]. We analyse dynamics of the $SU(1,1)$ coherent states, generated by a Hamiltonian given in terms of the bosonic creation and annihilation operators $a^\dagger$, a as

$$H = \tfrac{1}{2}\hbar\omega\big(a^\dagger a + aa^\dagger\big) + \tfrac{1}{2}\hbar\chi(t)a^{\dagger 2} + \tfrac{1}{2}\hbar\chi^*(t)a^2. \tag{1.1}$$

Time evolution of the coherent states will be visualized by phase space trajectories, which, as we shall see, can be divided into two classes. To the first class belong those trajectories which occupy a bounded region in the Lobachevskii plane and never approach the unit circle (compact trajectories). Trajectories which after sufficiently long time become arbitrarily close to the unit circle (but do not reach it) belong to the second class of noncompact trajectories. We shall investigate phase space dynamics for a 'pumping field' $\chi(t)$ of the type

$$\chi(t) = c(t)\mathrm{e}^{-\mathrm{i}\alpha(t)} \tag{1.2}$$

with

$$c(t) = \chi_0 \cos\omega_1 t \qquad \alpha(t) = \omega_2 t. \tag{1.3}$$

Our main purpose is to study the type of dynamics (compact or noncompact) depending on the values of the frequencies ω_1, ω_2 and the amplitude χ_0.

A very efficient tool for the description of the dynamics of the time-dependent harmonic oscillator can be provided by the method of unitary $SU(1,1)$ transformations with the time-dependent parameters [8]. The group transformation will be chosen to reduce the Hamiltonian, given originally as a combination of all three elements of Lie algebra, to only one generator, either compact or noncompact (the group has two compact and one noncompact generators). This method is used here to analyse the phase space dynamics of the $SU(1,1)$ coherent states. Dynamics of the transformed coherent state, driven by the reduced Hamiltonian, can then be found in a simple way. Applying later the inverse transformation one can determine the time evolution of the initial coherent state and trajectories in the phase space. The crucial point is that group transformations preserve coherence, so that the transformed state is also an $SU(1,1)$ coherent state, driven by the reduced, coherence preserving Hamiltonian. As will be seen in subsequent sections, the type of time dependence of the group parameters determining the appropriate group transformation reflects the character of phase space trajectories. For compact trajectories the group parameters remain bound and oscillate, whereas in the noncompact case their absolute values increase indefinitely. Numerical analysis of the differential equations fulfilled by group parameters shows that also in the noncompact case the parameters may oscillate, but with increasing amplitude. In fact, these oscillations have rather a character of rapid, practically discontinous, jumps from positive to negative values with increasing absolute value.

The paper is organized as follows. In section 2 we describe the model of the generalized time-dependent harmonic oscillator under consideration and give explicit forms of the unitary transformations reducing the Hamiltonian to a single element of the $SU(1,1)$ Lie algebra. Section 3 is devoted to general discussion of the phase space dynamics of $SU(1,1)$ coherent states, based on the results of section 2 and transformation formulae of the phase space induced by the group transformations. In section 4 we discuss phase portraits of the $SU(1,1)$ coherent states, obtained by solving numerically equations for the trajectories and group parameters. A relation between the character of trajectories and behaviour of group parameters as functions of time will be clearly visible here. This section contains also qualitative analytic discussion of the coherent state orbits based on the Floquet theory, and of the singular behaviour of group parameters. Section 5 contains final remarks, and transformation formulae of group generators under group transformations are summarized in appendix A. In appendix B a relation is shown between parameters of the $SU(1,1)$ coherent state and parameters of a group transformation diagonalizing the Hamiltonian.

2. The model and reduction of the Hamiltonian

We consider a model physical system described by the Hamiltonian (1.1) which can be written as a linear combination of the $SU(1,1)$ group generators with time-dependent coefficients

$$H = 2\hbar\omega K_0 + \hbar\chi(t)K_+ + \hbar\chi^*(t)K_- \tag{2.1}$$

where the operators K_0, K_+ and K_- have the following representation in terms of the usual creation and annihilation operators of the Weyl–Heisenberg group [16, 20]:

$$K_0 = \tfrac{1}{4}\left(a^\dagger a + aa^\dagger\right) \qquad K_+ = \tfrac{1}{2}a^{\dagger 2} \qquad K_- = \tfrac{1}{2}a^2. \tag{2.2}$$

This representation corresponds to the values of the Bargmann index k equal to $1/4$ and $3/4$ [21]. In terms of the photon number states $k = 1/4$ corresponds to the subspace with even photon number and $k = 3/4$ to odd photon number [16]. The Hamiltonian (2.1) can be expressed in terms of the Hermitian generators

$$K_0 \qquad K_1 = \tfrac{1}{2}(K_+ + K_-) \qquad K_2 = -\frac{\mathrm{i}}{2}(K_+ - K_-) \tag{2.3}$$

as

$$H = 2\hbar\omega K_0 + 2\hbar c(t)\cos\alpha(t)K_1 + 2\hbar c(t)\sin\alpha(t)K_2 \tag{2.4}$$

where the coupling has been assumed to have the form (1.2)

$$\chi(t) = c(t)\mathrm{e}^{-\mathrm{i}\alpha(t)}. \tag{2.5}$$

Hermitian generators fulfill well known commutation rules

$$[K_1, K_2] = -\mathrm{i}K_0 \qquad [K_2, K_0] = \mathrm{i}K_1 \qquad [K_0, K_1] = \mathrm{i}K_2. \tag{2.6}$$

As the $SU(1,1)$ group is noncompact, its unitary representations are infinite dimensional. We shall also use the non-Hermitian two-dimensional representation of the group with generators given in terms of Pauli matrices [14]

$$\kappa_1 = \frac{\mathrm{i}}{2}\sigma_2 \qquad \kappa_2 = -\frac{\mathrm{i}}{2}\sigma_1 \qquad \kappa_0 = \tfrac{1}{2}\sigma_3. \tag{2.7}$$

Classical dynamics of $SU(1,1)$ coherent states [16] is defined as the dynamics generated by the classical Hamiltonian

$$H_{\mathrm{cl}}\left(\xi, \xi^*\right) = \langle\xi|H|\xi\rangle \tag{2.8}$$

where $|\xi\rangle$ is the $SU(1,1)$ coherent state [13, 22], having the following expansion in terms of the basis states $|n,k\rangle$ with given Bargmann index k:

$$|\xi, k\rangle = \left(1 - |\xi|^2\right)^k \sum_{n=0}^{\infty}\left[\frac{\Gamma(n+2k)}{n!\Gamma(2k)}\right]^{1/2} \xi^n |n,k\rangle. \tag{2.9}$$

These states can be obtained by action of an appropriate group element on the state $|0,k\rangle$

$$|\xi,k\rangle = \exp(zK_+ - z^*K_-)|0,k\rangle \tag{2.10}$$

where $z = -(\theta/2)e^{-i\phi}$, $\xi = -\tanh(\theta/2)e^{-i\phi}$. The parameters θ and ϕ, with ranges $-\infty < \theta < \infty$, $0 \leqslant \phi \leqslant 2\pi$, parametrize the group manifold consisting of two unconnected hyperboloids [11]. The group manifold plays an important role in the classical description of group dynamics since orbits of the pseudospin vector [11,23] lie on this manifold. Dynamics of the pseudospin (Bloch) vector, given as an expectation value of the generators, is closely related to orbits of the coherent states in the Lobachevskii plane by means of the stereographic image of the group manifold [11]. The equation of motion of the coherent state parameter ξ is equivalent to the Schrödinger equation for the time evolution operaror $S(t)$

$$i\hbar\dot{S(t)} = H(t)S(t) \tag{2.11}$$

with $S(t)$ parametrized as

$$S(t) = e^{\xi K_+}e^{\delta K_0}e^{-\xi^* K_-}e^{igK_0} \tag{2.12}$$

in an analogous way as has been done recently in the case of the $SU(2)$ group of the two-level system [24]. The remaining parameters in (2.12) fulfill $\delta = -2\ln(1-|\xi|^2)$ and $\dot{g} = -2\omega - 2\mathrm{Re}\,(\chi\xi^*)$. It follows from the algebra of group generators that the first three factors in (2.12) are the same as the displacement operator in (2.10), so the time displacement operator (2.12) generates, up to a phase factor, the coherent state $|\xi\rangle$ from the state $|0,k\rangle$. It follows further from (2.11) that the parameter ξ fulfills the same time evolution equation as (2.15) below, which follows from the classical Hamiltonian (2.8). The same classical representation is a natural consequence of the path-integral approach to the $SU(1,1)$ coherent states [13]. Another classical representation can be obtained with the use of the bosonic coherent states, as has been done for example in [17].

The time dependence of the parameter ξ in the classical dynamics generated by (2.8) is described by the first-order differential equation [13]

$$\dot{\xi} = \left\{\xi, H_{\mathrm{cl}}\left(\xi,\xi^*\right)\right\} \tag{2.13}$$

where the $SU(1,1)$ Poisson bracket is defined as

$$\{A,B\} = \frac{(1-|\xi|^2)^2}{2ik\hbar}\left(\frac{\partial A}{\partial\xi}\frac{\partial B}{\partial\xi^*} - \frac{\partial A}{\partial\xi^*}\frac{\partial B}{\partial\xi}\right). \tag{2.14}$$

The expectation value (2.8) can be calculated using general formulae obtained by Lisowski [26], and together with (2.13) and (2.14) this leads to the Riccati type equation for ξ

$$\dot{\xi} = -2i\omega\xi - i\chi^*\xi^2 - i\chi. \tag{2.15}$$

This equation can be solved by quadratures for constant χ and for χ of the form $\chi(t) = c(t)\exp(-2i\omega t)$ [12,27]. In this last case we substitute $\xi = \zeta\exp(-2i\omega t)$ and obtain for ζ

$$\dot{\zeta} = -ic(t)\big(\zeta^2+1\big) \tag{2.16}$$

which can be solved by separation of variables. This type of time dependence of χ corresponds to the degenerate parametric amplifier with time-dependent amplitude of the pumping field.

To analyse the classical phase space dynamics of $SU(1,1)$ coherent states with time-dependent coupling χ we shall use unitary group transformations diagonalizing the Hamiltonian. The time evolution of a quantum state is described by the Schrödinger equation

$$i\hbar\partial_t|\psi\rangle = H(t)|\psi\rangle. \tag{2.17}$$

By performing a time-dependent unitary transformation of the state $|\psi\rangle$:

$$|\phi\rangle = U(t)|\psi\rangle \tag{2.18}$$

we can write (2.17) in an equivalent form

$$\mathrm{i}\hbar\partial_t|\phi\rangle = H_1(t)|\phi\rangle \tag{2.19}$$

where the transformed Hamiltonian reads

$$H_1 = UHU^\dagger - \mathrm{i}\hbar U\dot{U}^\dagger. \tag{2.20}$$

We seek a group transformation $U(t)$ for which the transformed Hamiltonian contains only the compact generator K_0 [8]. First we perform the unitary transformation

$$U_0(t) = \mathrm{e}^{-\mathrm{i}\alpha(t)K_0} \tag{2.21}$$

which, upon using transformation formulae given in the appendix, transforms the Hamiltonian (2.4) to

$$H' = U_0HU_0^\dagger - \mathrm{i}\hbar U_0\dot{U}_0^\dagger = 2\hbar\left(\omega - \frac{\dot\alpha}{2}\right)K_0 + 2\hbar c(t)K_1. \tag{2.22}$$

Next we apply

$$U_1 = \mathrm{e}^{\mathrm{i}\varphi K_1}\mathrm{e}^{\mathrm{i}\beta K_2} \tag{2.23}$$

with time-dependent parameters φ and β. This gives

$$\begin{aligned}
H_1 = U_1H'U_1^\dagger - \mathrm{i}\hbar U_1\dot{U}_1^\dagger \\
&= 2\hbar K_0\left[\left(\omega - \frac{\dot\alpha}{2}\right)\cosh\beta - c\sinh\beta\right]\cosh\varphi \\
&\quad + 2\hbar K_1\left[-\left(\omega - \frac{\dot\alpha}{2}\right)\sinh\beta + c\cosh\beta\right] \\
&\quad + 2\hbar K_2\left[\left(\omega - \frac{\dot\alpha}{2}\right)\cosh\beta - c\sinh\beta\right]\sinh\varphi \\
&\quad - \hbar\dot\beta\,(K_0\sinh\varphi + K_2\cosh\varphi) - \hbar\dot\varphi K_1
\end{aligned} \tag{2.24}$$

where again the transformation formulae for the generators were used. The coefficients of noncompact generators K_1 and K_2 in (2.24) vanish if the transformation parameters φ and β fulfill a system of two coupled differential equations

$$\dot\varphi = -2\left(\omega - \frac{\dot\alpha}{2}\right)\sinh\beta + 2c\cosh\beta \tag{2.25a}$$

$$\dot\beta = \left[2\left(\omega - \frac{\dot\alpha}{2}\right)\cosh\beta - 2c\sinh\beta\right]\tanh\varphi. \tag{2.25b}$$

The transformed Hamiltonian H_1 has the form

$$H_1 = 2\hbar\Omega(t)K_0 \tag{2.26}$$

where

$$\Omega(t) = \frac{\left(\omega - \frac{\dot\alpha}{2}\right)\cosh\beta - c\sinh\beta}{\cosh\varphi}. \tag{2.27}$$

We shall also use an approach in which the Hamiltonian (2.4) is reduced to the form in which only the noncompact operator K_1 is left. The corresponding unitary transformation is given by

$$U = \mathrm{e}^{\mathrm{i}\mu K_2}\mathrm{e}^{\mathrm{i}\gamma K_0} \tag{2.28}$$

with time-dependent parameters μ and γ. This gives

$$
\begin{aligned}
H_2 &= UH'U^\dagger - \mathrm{i}\hbar U\dot{U}^\dagger \\
&= 2\hbar K_0\left[\left(\omega - \frac{\dot{\alpha}}{2}\right)\cosh\mu - c\cos\gamma\sinh\mu\right] \\
&\quad + 2\hbar K_1\left[-\left(\omega - \frac{\dot{\alpha}}{2}\right)\sinh\mu + c\cos\gamma\cosh\mu\right] \\
&\quad - 2\hbar c\sin\gamma K_2 - \hbar\dot{\gamma}\,(K_0\cosh\mu - K_1\sinh\mu) - \hbar\dot{\mu}K_2.
\end{aligned}
\tag{2.29}
$$

The coefficients of K_0 and K_2 vanish when μ and γ fulfill

$$\dot{\mu} = -2c\sin\gamma \tag{2.30a}$$

$$\dot{\gamma} = 2\left(\omega - \frac{\dot{\alpha}}{2}\right) - 2c\cos\gamma\tanh\mu. \tag{2.30b}$$

The transformed Hamiltonian has now the form

$$H_2 = 2\hbar\Gamma(t)K_1 \tag{2.31}$$

where

$$\Gamma(t) = c(t)\frac{\cos\gamma}{\cosh\mu}. \tag{2.32}$$

3. Dynamics of the $SU(1,1)$ phase space

Solutions of equation (2.15) can be divided into two classes. To the first class belong those solutions for which

$$|\xi| \leqslant a < 1 \tag{3.1}$$

and trajectories $\xi(t)$ in the Lobachevskii plane remain in a compact region inside the unit circle. For the solutions from the second class

$$\lim_{t\to\infty} |\xi| = 1 \tag{3.2}$$

the phase space trajectories approach asymptotically the unit circle, but never reach it. In this case the trajectories occupy a noncompact region, i.e. the interior of the unit circle without boundary. These two classes of trajectories can be illustrated by an example solution of (2.15) with constant $c > 0$ and $\alpha = \omega_2 t$ with $\omega_2 < 2\omega$. In this case we have

$$\xi(t) = \mathrm{e}^{-\mathrm{i}\omega_2 t}\frac{\xi_1(\xi_0 - \xi_2) - \xi_2(\xi_0 - \xi_1)\mathrm{e}^{-2\mathrm{i}\Omega t}}{\xi_0 - \xi_2 - (\xi_0 - \xi_1)\mathrm{e}^{-2\mathrm{i}\Omega t}} \tag{3.3}$$

where ξ_0 is the initial value of ξ and

$$\xi_1 = -c^{-1}\left[\omega - \frac{\omega_2}{2} - \sqrt{\left(\omega - \frac{\omega_2}{2}\right)^2 - c^2}\right] \tag{3.4a}$$

$$\xi_2 = -c^{-1}\left[\omega - \frac{\omega_2}{2} + \sqrt{\left(\omega - \frac{\omega_2}{2}\right)^2 - c^2}\right] \tag{3.4b}$$

$$\Omega = \sqrt{\left(\omega - \frac{\omega_2}{2}\right)^2 - c^2}. \tag{3.4c}$$

For $c < \omega - \omega_2/2$ the frequency Ω is real and $\xi(t)$ remains in a compact region inside the unit circle. The dynamics of the system is compact. If, however, $c > \omega - \omega_2/2$, the frequency Ω is purely imaginary, and for positive $\mathrm{Im}\,\Omega$ we have for $t \to \infty$

$$\xi(t) \to \mathrm{e}^{-\mathrm{i}\omega_2 t}\xi_2. \tag{3.5}$$

It can be seen from (3.4*b*) that in this case ξ_2 is complex and $|\xi_2| = 1$, so that $\xi(t)$ approaches the unit circle revolving around it with frequency ω_2. The dynamics is now noncompact.

On the level of the group transformations equations (2.25) for constant c and $\alpha = \omega_2 t$ have time-independent solutions given by

$$\varphi = 0 \qquad \tanh\beta = \frac{c}{\omega - \omega_2/2} \tag{3.6}$$

and the diagonalized Hamiltonian (2.26) can be written as

$$H_1 = 2\hbar\left(\omega - \frac{\omega_2}{2}\right)\left[1 - \frac{c^2}{(\omega - \omega_2/2)^2}\right]^{1/2} K_0. \tag{3.7}$$

Note that H_1 given by (3.7) becomes non-Hermitian for $c > \omega - \omega_2/2$. We see that for compact dynamics of $SU(1,1)$ coherent states the diagonalized Hamiltonian is expressed by the compact generator K_0. On the other hand equations (2.30) have time-independent solutions of the form

$$\gamma = 0 \qquad \tanh\mu = \frac{\omega - \omega_2/2}{c} \tag{3.8}$$

and then the diagonalized Hamiltonian (2.31) reads

$$H_2 = 2\hbar c\left[1 - \frac{(\omega - \omega_2/2)^2}{c^2}\right]^{1/2} K_1 \tag{3.9}$$

and is Hermitian for $c > \omega - \omega_2/2$. For noncompact dynamics the diagonalized Hamiltonian contains the noncompact generator K_1.

For time-dependent c and α also the group parameters (φ, β) or (μ, γ) depend on time. Note, however, that time-dependent group parameters can be also used in the just-discussed case of constant c, as equations (2.25) and (2.30) have then also solutions varying in time. With time-dependent group parameters either (2.26) or (2.31) can be used as the diagonalized Hamiltonian.

To express the phase space parameter of the generalized coherent state, $\xi(t)$, in terms of (φ, β) or (μ, γ) we make use of the two-dimensional non-Hermitian representation of the group generators (2.7). In the two-dimensional realization any group element g can be written as [14][1].

$$g = \begin{bmatrix} a & b \\ b^* & a^* \end{bmatrix} \qquad |a|^2 - |b|^2 = 1. \tag{3.10}$$

The corresponding group transformation $V(g)$ in the unitary representation with Bargmann index k acts on the coherent state $|\xi\rangle$ in the following way:

$$V(g)|\xi\rangle = \mathrm{e}^{\mathrm{i}\delta}|\zeta\rangle \tag{3.11}$$

where

$$\zeta = \frac{a\xi + b}{b^*\xi + a^*} \qquad \delta = \arg(a - b\xi). \tag{3.12}$$

Using (2.7) and properties of the Pauli matrices we can see that the transformation diagonalizing the Hamiltonian to the compact form (2.26)

$$V(g) = U_1 U_0 \tag{3.13}$$

[1] $SU(1,1)$ is a group of 2×2 pseudounitary unimodular matrices leaving invariant the quadratic form $|z_1|^2 - |z_2|^2$.

with U_0 and U_1 given by (2.21) and (2.23), respectively, corresponds to the group element of the form (3.10) with

$$a = \left(\cosh\frac{\varphi}{2}\cosh\frac{\beta}{2} + \mathrm{i}\sinh\frac{\varphi}{2}\sinh\frac{\beta}{2}\right)\mathrm{e}^{\mathrm{i}\alpha/2} \tag{3.14a}$$

$$b = \left(\cosh\frac{\varphi}{2}\sinh\frac{\beta}{2} + \mathrm{i}\sinh\frac{\varphi}{2}\cosh\frac{\beta}{2}\right)\mathrm{e}^{-\mathrm{i}\alpha/2}. \tag{3.14b}$$

The classical dynamics of the transformed parameter ζ is generated by the Hamiltonian (cf (2.26))

$$H_{\mathrm{cl}}\left(\zeta,\zeta^*\right) = \langle\zeta|H_1|\zeta\rangle = 2\hbar\Omega(t)\langle\zeta|K_0|\zeta\rangle. \tag{3.15}$$

Using

$$\langle\zeta|K_0|\zeta\rangle = k\frac{1+|\zeta|^2}{1-|\zeta|^2} \tag{3.16}$$

and the general equations of motion (2.13) and (2.14) we find

$$\dot{\zeta} = -2\mathrm{i}\Omega(t)\zeta \tag{3.17}$$

with a simple solution

$$\zeta(t) = \exp\left[-2\mathrm{i}\int_0^t \Omega(t')\,\mathrm{d}t'\right]\zeta_0. \tag{3.18}$$

With the initial conditions for the group parameters $\varphi(0) = 0$ and $\beta(0) = 0$ we see that the initial value ζ_0 is equal to the initial value ξ_0 of the original parameter ξ. Inverting (3.12) and using (3.18) we obtain

$$\xi(t) = \frac{b(t) - a^*(t)\zeta(t)}{b^*(t)\zeta(t) - a(t)}. \tag{3.19}$$

We calculate further

$$\nu = 1 - |\xi|^2 = \frac{1-|\zeta|^2}{|a|^2 + |b|^2|\zeta|^2 - 2\mathrm{Re}\,(ab\zeta^*)}. \tag{3.20}$$

For compact dynamics $\nu(t)$ does not approach zero, and for noncompact dynamics

$$\lim_{t\to\infty}\nu(t) = 0. \tag{3.21}$$

Taking that

$$|\zeta| = |\zeta_0| \tag{3.22}$$

and looking at (3.14) and (3.20) we see that $\nu(t)$ does not vanish if the group parameters φ and β remain finite during time evolution. On the other hand, (3.20) can be written as

$$\nu = \frac{1-|\zeta|^2}{\cosh^2(\varphi/2)\cosh^2(\beta/2)}\lambda(t) \tag{3.23}$$

where $\lambda(t)$ is always finite, no matter what the values of φ and β are. Equation (3.23) shows that the dynamics is noncompact whenever the absolute value of one of the group parameters (or both) increases indefinitely with increasing time. We shall see later (section 4) that the time variation of the group parameters is usually nonmonotonic, and consists rather of rapid 'jumps' from large positive to small negative values (with large absolute value).

If the unitary group transformation (2.28) is used the relation between original and transformed phase parameters, ξ and ζ respectively, reads

$$\xi = \mathrm{e}^{-\mathrm{i}(\alpha+\gamma)}\frac{\sinh(\mu/2) - \zeta\cosh(\mu/2)}{\zeta\sinh(\mu/2) - \cosh(\mu/2)} \tag{3.24}$$

and the equation of motion for ζ has the form

$$\dot{\zeta} = \{\zeta, \langle\zeta|H_2|\zeta\rangle\} \tag{3.25}$$

where H_2 is given by (2.31) and (2.32). Using

$$\langle\zeta|K_1|\zeta\rangle = k\frac{\zeta + \zeta^*}{1 - |\zeta|^2} \tag{3.26}$$

we obtain

$$\dot{\zeta} = -\mathrm{i}\Gamma(t)(1 + \zeta^2). \tag{3.27}$$

This equation has a solution

$$\zeta = \frac{\zeta_0 \cosh[f(t)] - \mathrm{i}\sinh[f(t)]}{\cosh[f(t)] + \mathrm{i}\zeta_0 \sinh[f(t)]} \tag{3.28}$$

where

$$f(t) = \int_0^t \Gamma(t')\,\mathrm{d}t' \tag{3.29}$$

and ζ_0 is the initial value of ζ, and is equal to the initial value of the original phase space parameter ξ_0. From (3.24) we obtain

$$1 - |\xi|^2 = \frac{1 - |\zeta|^2}{|\zeta \sinh(\mu/2) - \cosh(\mu/2)|^2}. \tag{3.30}$$

Equation (3.30) shows that the dynamics is noncompact when either

$$\lim_{t\to\infty} |\zeta| = 1 \tag{3.31}$$

or μ increases indefinitely with increasing time. It follows further from (3.28) that (3.31) is true only if

$$\lim_{t\to\infty} |f(t)| = \infty. \tag{3.32}$$

However, equation (2.32) suggests that this could never happen, since the integrand $\Gamma(t)$ of (3.29) can either oscillate in time, due to the numerator of (2.32), or go to zero if the group parameter μ in the denominator goes to infinity. Therefore, the only criterion of the noncompact dynamics is an indefinite increase of the group parameter μ. Note that for both forms of the diagonalized Hamiltonian, (2.26) and (2.31), the dynamics of the transformed group parameter ζ is compact. Only after applying the group transformation from ζ to ξ can one see the true character of the dynamics in the original Lobachevskii plane.

An alternative description of the dynamics of the $SU(1,1)$ phase space can be given by the time evolution operator. The time evolution of the state (cf (2.21))

$$|\xi'; t\rangle = \exp[\mathrm{i}\alpha(t)K_0]|\xi; t\rangle \tag{3.33}$$

is generated by the Hamiltonian H' (2.22), so the time evolution operator fulfills

$$\mathrm{i}\hbar\frac{\partial S(t)}{\partial t} = H'(t)S(t) \tag{3.34}$$

with the initial condition $S(0) = 1$. To solve (3.34) we use a method similar to that developed by Dattoli *et al* [28] and [18], based on the Wei–Norman [29] and Magnus [30] rigorous algebraic procedure. We are looking for the evolution operator in the form

$$S(t) = \exp[-\mathrm{i}h(t)K_2]\exp[-\mathrm{i}g(t)K_1]\exp[-\mathrm{i}k(t)K_0]. \tag{3.35}$$

Substituting (3.35) into the left-hand side of (3.34) and (2.22) into the right-hand side, and comparing coefficients at the same generators on both sides, we obtain a system of differential equations for the parameters h, g and k:

$$\dot{g} = -2\left(\omega - \frac{\dot{\alpha}}{2}\right)\sinh h + 2c\cosh h \tag{3.36a}$$

$$\dot{h} = \left[2\left(\omega - \frac{\dot{\alpha}}{2}\right)\cosh h - 2c\sinh h\right]\tanh g \tag{3.36b}$$

$$\dot{k} = 2\frac{\left(\omega - \frac{\dot{\alpha}}{2}\right)\cosh h - c\sinh h}{\cosh g}. \tag{3.36c}$$

Comparing (3.36) with (2.25) we see that g and h can be identified with φ and β respectively, and from (3.36c) and (2.27) it follows that

$$\dot{k} = 2\Omega(t). \tag{3.37}$$

Therefore

$$S(t) = \mathrm{e}^{-\mathrm{i}\beta K_2}\mathrm{e}^{-\mathrm{i}\varphi K_1}\exp\left[-2\mathrm{i}\int_0^t \Omega(t')\,\mathrm{d}t'\,K_0\right]. \tag{3.38}$$

Expression (3.38) for the time evolution operator can also be calculated from the results of section 2. The transformed state

$$|\zeta\rangle = \mathrm{e}^{\mathrm{i}\varphi K_1}\mathrm{e}^{\mathrm{i}\beta K_2}|\xi'\rangle \tag{3.39}$$

evolves in time according to

$$|\zeta\rangle = \exp\left[-2\mathrm{i}\int_0^t \Omega(t')\,\mathrm{d}t'\,K_0\right]|\xi_0\rangle \tag{3.40}$$

where $\xi_0 = \zeta_0$ is the initial value of the phase space parameter. We have therefore

$$|\xi', t\rangle = \mathrm{e}^{-\mathrm{i}\beta K_2}\mathrm{e}^{-\mathrm{i}\varphi K_1}\exp\left[-2\mathrm{i}\int_0^t \Omega(t')\,\mathrm{d}t'\right]|\xi_0\rangle \tag{3.41}$$

in agreement with (3.38). In the same way one may obtain the time evolution operator using the unitary transformation (2.28). The result is

$$S(t) = \mathrm{e}^{-\mathrm{i}\gamma(t)K_0}\mathrm{e}^{-\mathrm{i}\mu(t)K_2}\exp\left[-2\mathrm{i}\int_0^t \Gamma(t')\,\mathrm{d}t'\,K_1\right] \tag{3.42}$$

where γ and μ are solutions of (2.30) and Γ is given by (2.32).

4. Phase portraits of the $SU(1,1)$ coherent states

We shall discuss now examples of the phase space dynamics of $SU(1,1)$ coherent states, and show explicitly the transition from one type of dynamics to the other. We assume that both the amplitude and the phase of the coupling function χ depend on time and the specific form of $\chi(t)$ is given by

$$\chi(t) = \chi_0\cos\omega_1 t\,\mathrm{e}^{-\mathrm{i}\omega_2 t}. \tag{4.1}$$

This corresponds to a pumping field of frequency ω_2 and modulated amplitude. We have seen in the previous section that for constant amplitude the dynamics is compact when $\chi_0 < \omega - \omega_2/2$ and noncompact otherwise. The time dependence of the amplitude may change this behaviour significantly, but for a slowly varying amplitude (ω_1 small) we may expect a similar behaviour as for a constant amplitude of similar magnitude, and only larger frequencies ω_1 may lead to significant differences in comparison with the constant-amplitude case. The type of dynamics will be further illustrated by looking both at the shape of the ξ-trajectories and the behaviour of the parameters of appropriate group transformations (2.23) or (2.28).

Before coming to numerical examples we shall perform a partly qualitative analytical analysis of equations (2.15) and (2.25). The analysis of the Riccati equation (2.15) will be based on the Floquet theory of linear differential equations with periodic coefficients ([24] and references therein). The idea is to replace the nonlinear first-order Riccati equation (2.15) by a linear second-order equation [24, 25]. With this aim in mind we perform a chain of transformations:

$$\xi = \zeta e^{-i\alpha(t)} \tag{4.2}$$

which is equivalent to (2.21), with ζ fulfilling

$$\dot{\zeta} = -i(1+\zeta^2)\chi_0 \cos\omega_1 t + i(\omega_2 - 2\omega)\zeta. \tag{4.3}$$

In the next step a conformal transformation is performed

$$\xi = i\frac{1-\eta}{1+\eta} \tag{4.4}$$

leading also to the Riccati equation with constant coefficient at the quadratic term

$$\dot{\eta} = \eta\chi_0 \cos\omega_1 t - \frac{i}{2}(\omega_2 - 2\omega)(1-\eta^2). \tag{4.5}$$

The third transformation reads

$$\eta = \frac{2i}{\omega_2 - 2\omega}\frac{\dot{q}}{q} \tag{4.6}$$

where q is a new unknown function fulfilling the second-order linear differential equation, which in terms of the dimensionless variable $x = \omega_1 t$ can be written as

$$q'' - 2\gamma_0 \cos x \cdot q' + \frac{\epsilon^2}{4} q = 0 \tag{4.7}$$

where $\gamma_0 = \chi_0/\omega_1$, $\epsilon = (\omega_2 - 2\omega)/\omega_1$ and the prime denotes differentiation with respect to x, and $q(0) = 1$, $q'(0) = -i\epsilon/2$, consistent with $\xi(0) = 0$. This is an equation with periodic coefficients and to analyse properties of the solutions one can apply the Floquet theory [25]. We consider two real fundamental solutions $u(x)$ and $v(x)$ such that $u(0) = 1$, $u'(0) = 0$ and $v(0) = 0$, $v'(0) = 1$. There exists a solution $f(x)$ to (4.7) fulfilling the Floquet condition $f(x + 2\pi) = \lambda f(x)$, where λ is a solution of the characteristic equation

$$\lambda^2 + [u(2\pi) + v'(2\pi)]\lambda + 1 = 0. \tag{4.8}$$

If $|u(2\pi)+v'(2\pi)| < 2$ there are two complex mutually conjugate solutions with unit modulus, $|\lambda_{1,2}| = 1$. If $|u(2\pi) + v'(2\pi)| > 2$ there are two real solutions, one with the absolute value bigger than unity and the other with the absolute value smaller than unity. A general solution to (4.7) has the form

$$q(x) = \exp\left(\frac{x}{2\pi}\ln\lambda_1\right) F_1(x) + \exp\left(\frac{x}{2\pi}\ln\lambda_2\right) F_2(x) \tag{4.9}$$

where $F_1(x)$ and $F_2(x)$ are periodic functions which can be expanded in the Fourier series. The case of complex roots of (4.8) results in the oscillatory character of the solutions to (4.7) and, in the language of the coherent state orbits, to compact dynamics with the ξ trajectories confined to a compact region inside a unit circle. If, however, the solutions of (4.8) are real then taking $|\lambda_1| > 1$ we observe that first term in (4.9) oscillates with increasing amplitude, whereas the second one tends to zero for $x \to \infty$, so that for large x only the first term survives. This leads to unstable solutions of (4.7) corresponding to noncompact dynamics, where ξ-orbits approach the unit circle.

We performed three types of calculation of the ξ-orbits: by direct numerical solution of the Riccati equation (2.15), by solving the linear equation (4.7) numerically and then finding the orbits from (4.2)–(4.6) and, finally, calculating the $q(x)$-function analytically using a method similar to that described in [24] with the Floquet coefficients λ_i found numerically. In this last case the accuracy of the Floquet coefficients was tested by a consistency check, i.e. the counterpart of equation (6.13) in [24]. Not surprisingly, all three methods resulted in the same shapes of the ξ-orbits, and exactly the same time dependence of $1 - |\xi|^2$, $\mathrm{Re}\,\xi$, $\mathrm{Im}\,\xi$ etc.

Another interesting mathematical question refers to a singular behaviour of the group transformation parameters φ and β in the case of noncompact dynamics. It follows from (3.23) that these parameters should increase indefinitely, and it has been mentioned that variation in time of φ and β is nonmonotonic with rapid jumps from large positive to small negative values (with large absolute values). To examine solutions of (2.25) close to the singular point, where the parameter β is positive and large enough, we can approximate the hyperbolic sine and cosine by $\exp(\beta)/2$. We choose the initial value of time, t_0, close to the singular point with the corresponding values of β and φ equal to β_0 and φ_0 respectively. Equations (2.25) are approximated by

$$\dot{\varphi} = -\mathrm{e}^{\beta}\,[\Delta\omega - c(t)] \tag{4.10a}$$

$$\dot{\beta} = \mathrm{e}^{\beta}\,[\Delta\omega - c(t)]\tanh\varphi \tag{4.10b}$$

where $\Delta\omega = \omega - \omega_2/2$ and $c(t) = \chi_0\cos\omega_1 t$. Dividing equations (4.10a) and (4.10b) by each other and integrating by separation of variables we obtain

$$\cosh\varphi = A\mathrm{e}^{-\beta} \tag{4.11}$$

where

$$A = \mathrm{e}^{\beta_0}\cosh\varphi_0. \tag{4.12}$$

It follows from (4.11) and (4.12) that for $\beta > \beta_0$ $\varphi < \varphi_0$, i.e. the parameter φ decreases close to the singularity of β. Expressing $\tanh\varphi$ by β from (4.11), substituting into (4.10b) and integrating by separation of variables, we obtain $\beta(t)$ close to the singular point

$$\beta(t) = -\frac{1}{2}\ln\left\{\frac{1}{A^2} + \left[\mathrm{e}^{-\beta_0} - \Delta\omega(t - t_0) + \int_{t_0}^{t} c(\tau)\,\mathrm{d}\tau\right]^2\right\}. \tag{4.13}$$

For large β_0 and φ_0 the constant A is also large and A^{-2} is small. The logarithmic function in (4.13) acquires maximum absolute value for the value of $t = t_\mathrm{s}$ given as a solution of the equation

$$\mathrm{e}^{-\beta_0} - \Delta\omega(t_\mathrm{s} - t_0) + \int_{t_0}^{t_\mathrm{s}} c(\tau)\,\mathrm{d}\tau = 0. \tag{4.14}$$

Due to the very small value of A^{-2} the right-hand side of (4.13) is large, corresponding to singularity of the parameter β with cusplike behaviour clearly visible e.g. in figure 2(c).

To find $\varphi(t)$ close to singularity we substitute (4.13) into (4.10a) and integrate. This gives

$$\varphi(t) = \varphi_0 + \beta_0 - \ln 2 + \ln\left[u(t) + \sqrt{A^{-2} + u(t)^2}\right] \tag{4.15}$$

where we denoted for brevity

$$u(t) = \mathrm{e}^{-\beta_0} - \Delta\omega(t - t_0) + \int_{t_0}^{t} c(\tau)\,\mathrm{d}\tau. \tag{4.16}$$

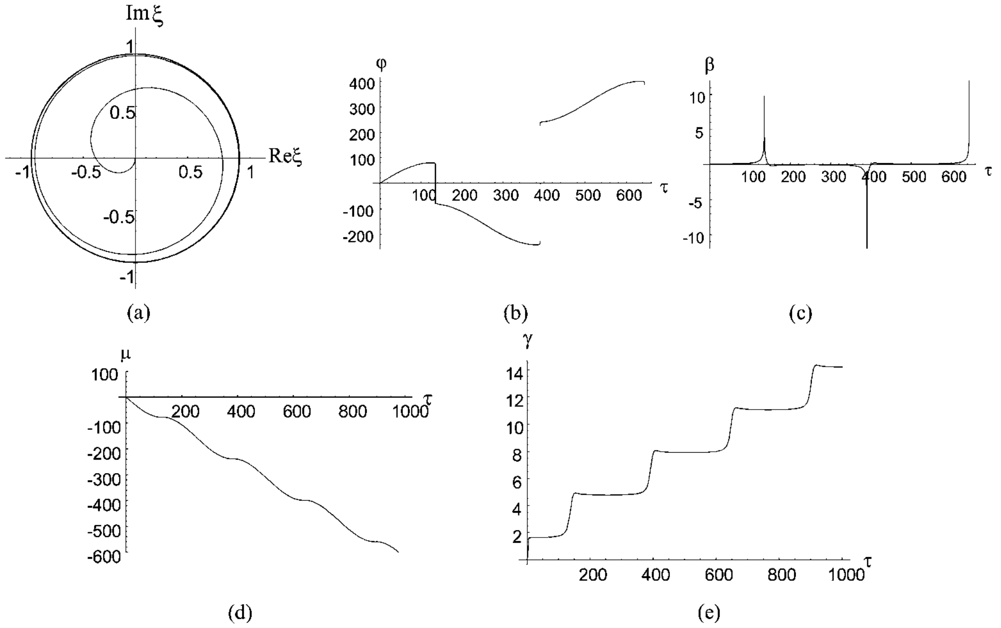

Figure 1. (*a*) Phase space trajectory, and (*b*)–(*e*) time dependence of group parameters φ, β, μ and γ in the noncompact case with $\chi_0 = 0.5, \omega_2 = 1.934$ and $\omega_1 = 0.0123$. Values of the Floquet coefficients are $\lambda_1 = -7.49 \times 10^{69}$ and $\lambda_2 = -1.35 \times 10^{-70}$. See text for more detailed description.

At the singular point of $\beta(t)$, i.e. for $t = t_s$ such that $u(t_s) = 0$ (cf (4.14)), the value of φ is

$$\varphi(t_s) = \varphi_0 + \beta_0 - \ln 2 - \ln A = 0 \tag{4.17}$$

(cf (4.12)) so that φ decreases very rapidly from φ_0 to zero. This can be seen e.g. in figure 3(*c*), where the vanishing value of φ corresponds to the peak value of β. For the values of time larger than the singular point t_s $u(t)$ is negative and quickly acquires absolute value much larger than A^{-2}. The parameter $\varphi(t)$ can be then approximated by

$$\varphi(t) = \varphi_0 + \beta_0 - \ln 2 - 2\ln A - \ln|u(t)| \approx -\ln A - \ln|u(t)|. \tag{4.18}$$

The time-dependent term in (4.18) is a slowly varying function of time with values small compared with remaining terms. Therefore, after passing the singular point $\varphi(t)$ is negative and slowly varying with large absolute value, until the next singular point is reached. Behaviour of this type can be seen for example in figure 3(*c*) or 6(*c*). At the next singular point the peak value of β is negative and φ jumps rapidly from negative to large positive values. This behaviour can be qualitatively analysed in a similar way as previously with the approximation to (2.25) appropriate for $\beta < 0$ and $|\beta| \gg 1$.

In the numerical examples we measure χ_0, ω_1 and ω_2 in units of the oscillator frequency ω, and the dimensionless time is $\tau = \omega t$. We choose first

$$\chi_0 = 0.5 \qquad \omega_2 = 1.934. \tag{4.19}$$

In the case of constant amplitude these values of the parameters lead to noncompact dynamics.

Figure 1(*a*) shows the phase space trajectory for small $\omega_1 = 0.0123$; we can see that the trajectory approaches the unit circle very quickly. The dependence of the group parameters φ and β on time is shown in figures 1(*b*) and (*c*) respectively. We can see that both parameters exhibit some sort of singular behaviour with rapid jumps of φ from positive to negative and back

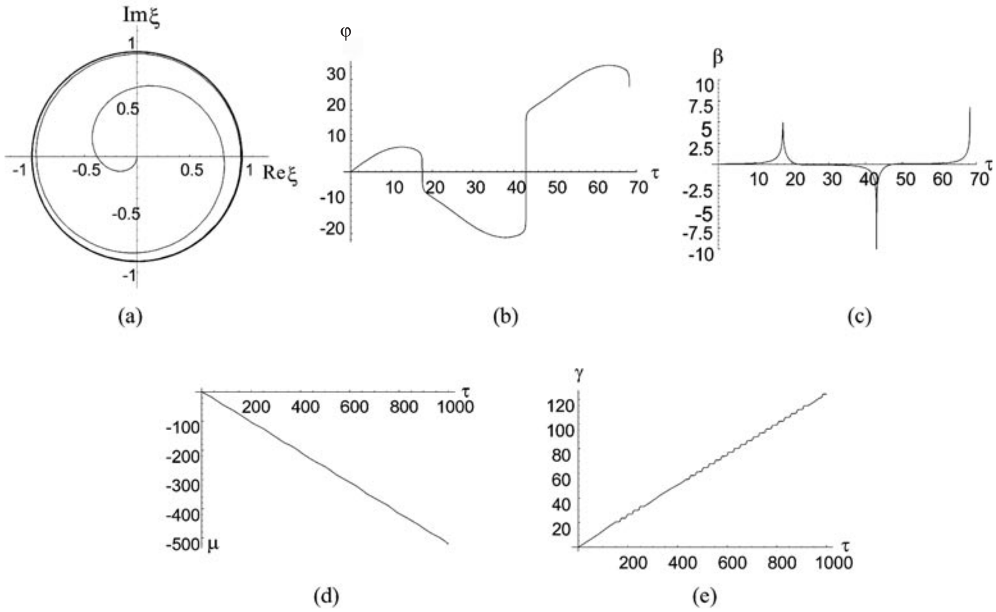

Figure 2. Same as figure 1 with $\omega_1 = 0.123$. Values of the Floquet coefficients are $\lambda_1 = -6.17 \times 10^5$ and $\lambda_2 = -1.62 \times 10^{-6}$.

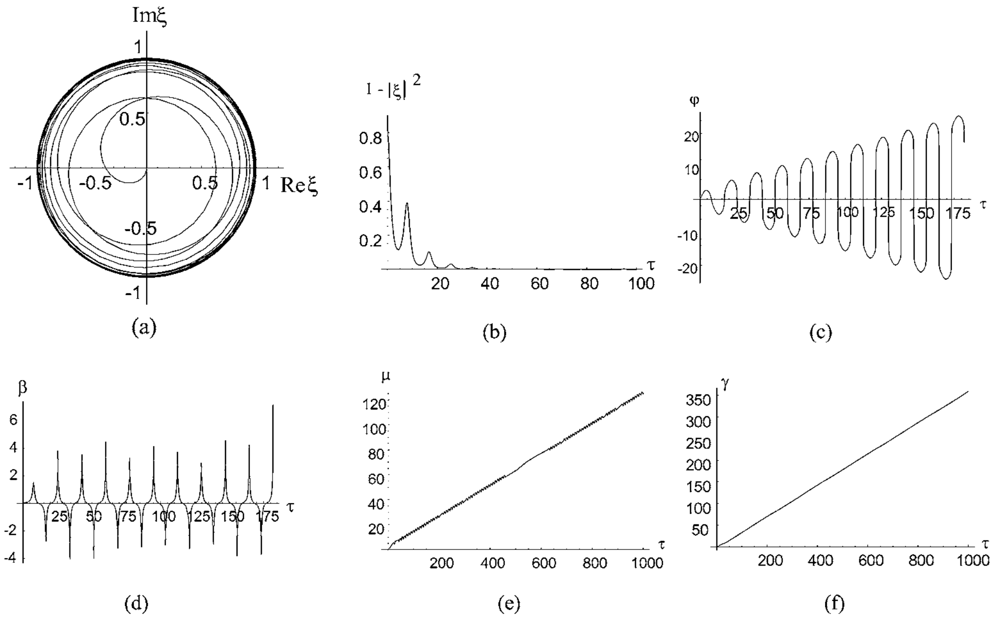

Figure 3. (*a*) Phase space trajectory, (*b*) $1 - |\xi|^2$ and (*c*)–(*f*) time dependence of group parameters φ, β, μ and γ in the noncompact case with $\chi_0 = 0.5$, $\omega_2 = 1.934$ and $\omega_1 = 0.362$. Values of the Floquet coefficients are $\lambda_1 = -2.91$ and $\lambda_2 = -0.34$.

to positive values, and cusplike singularities of β. Absolute values of φ increase after every ‘jump’ (this is even more visible in further examples of noncompact dynamics, figures 3(*c*)

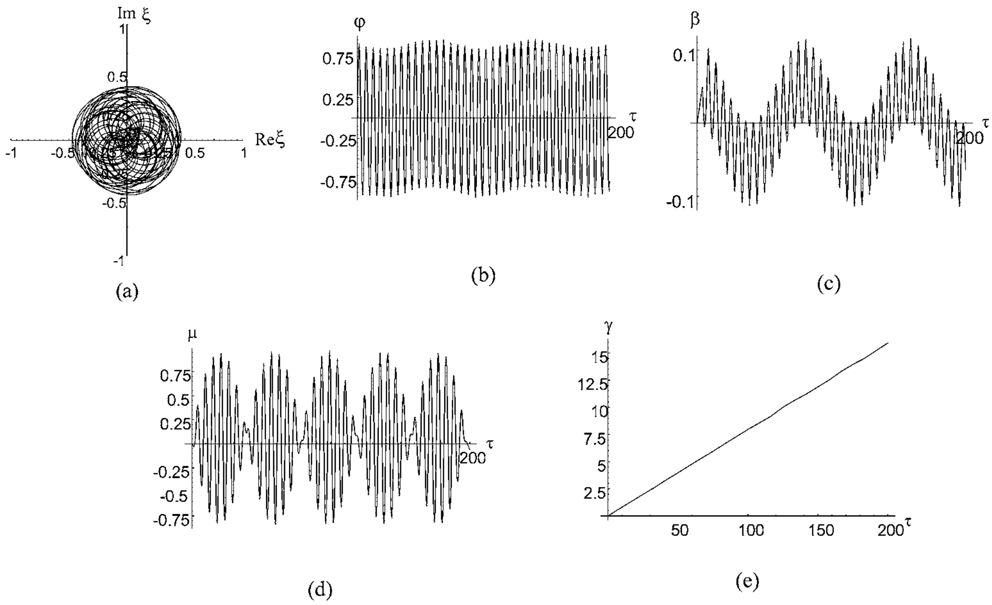

Figure 4. (*a*) Phase space trajectory and (*b*)–(*e*) time dependence of group parameters φ, β, μ and γ in the compact case with $\chi_0 = 0.5, \omega_2 = 1.934$ and $\omega_1 = 1.125$. Values of the Floquet coefficients are $\lambda_1 = 0.98 - 0.22\mathrm{i}$ and $\lambda_2 = 0.98 + 0.22\mathrm{i}$.

or 6(*c*)), and the peak values of β are relatively large. For this reason the right-hand side of (3.23) tends to zero in the long time run. The values of φ are large when β is small and *vice versa*—whenever φ passes through zero β reaches its extremum value at the cusp. In both cases there are regions of rapid variations of the group parameters difficult to handle numerically, so the time dependence of φ and β is shown in a shorter time interval than more regular and smooth parameters μ and γ (see below). As can be further seen from figure 1(*d*) the group parameter μ (cf (2.30) goes to minus infinity and the parameter γ, figure 1(*e*), increases with regions of slow variation separated by periods with large positive values of the derivative. However, both these parameters behave in a much more regular way than φ and β.

For ω_1 an order of magnitude smaller ($\omega_1 = 0.123$) we see from figure 2(*a*) that the dynamics is still noncompact. The behaviour of group parameters φ and β is shown in figures 2(*b*) and (*c*), where regions of rapid variations and sharp peaks are clearly visible. It is also clearly seen that the parameter β develops a series of sharp peaks during time evolution, as mentioned above. The peaks occur at the same values of time at which the parameter φ jumps rapidly. Parameters μ and γ show similar behaviour as in the previous case with a general tendency to increase their absolute values.

The phase space dynamics keeps its noncompact character up to values of ω_1 equal to approximately 0.365. For $\omega_1 = 0.362$ (figure 3(*a*)) trajectory approaches the unit circle in a long run, but may temporarily return to the internal region, as can be seen from the plot of $1 - |\xi|^2$ (figure 3(*b*)). Figures 3(*c*) and (*d*) show characteristic behaviour of the group parameters φ and β with rapid jumps of the former and cusplike behaviour of the latter, with peaks occurring again for the time values at which φ varies rapidly.

With increasing value of the frequency ω_1 the dynamics changes its character and becomes compact. This is illustrated in figure 4 obtained for $\omega_1 = 1.125$. The trajectory in the phase space is now confined to the internal region with a diameter smaller than unity, and does not

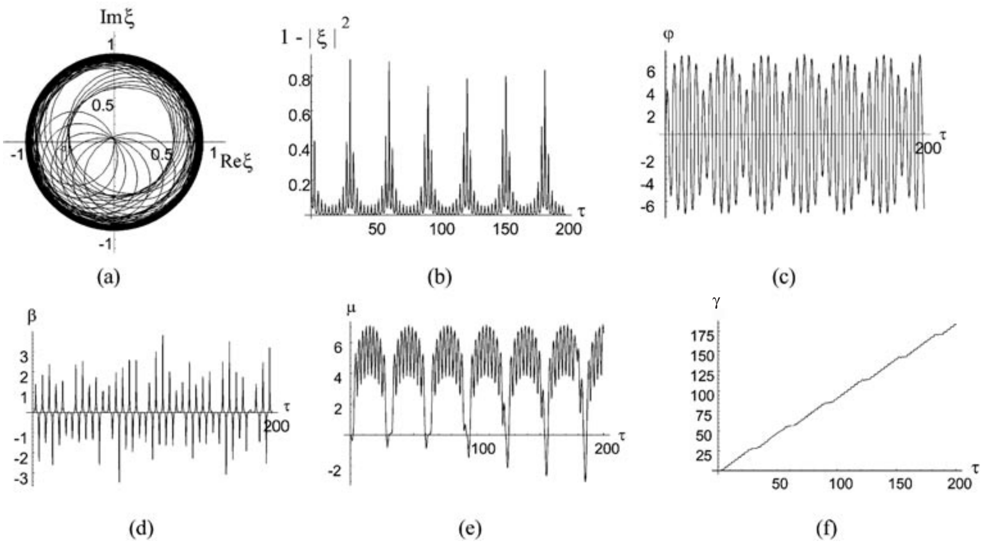

Figure 5. (a) Phase space trajectory, (b) $1 - |\xi|^2$ and (c)–(f) time dependence of group parameters φ, β, μ and γ in the transient case with $\chi_0 = 2.2, \omega_2 = 1.934$ and $\omega_1 = 1.125$. Values of the Floquet coefficients are $\lambda_1 = -0.86 - 0.51\mathrm{i}$ and $\lambda_2 = -0.86 + 0.51\mathrm{i}$.

approach the unit circle. The parameters φ and β remain bounded and exhibit periodic time dependence (figures 4(b) and (c)). The same is also true for the parameter μ (figure 4(d)), whereas γ increases to infinity.

We expect that a return to a noncompact type of dynamics should occur for increasing values of χ_0. Therefore, further numerical calculations were performed for fixed $\omega_1 = 1.125$ and $\omega_2 = 1.934$, and with gradually increasing values of χ_0. The dynamics remains compact for χ_0 increasing up to $\chi_0 \approx 2.2$ (figure 5). It might seem from figure 5(a) that the phase space trajectory approaches the unit circle, but the plot of $1 - |\xi|^2$ (figure 5(b)) shows periodic deviations from small values, which indicates that trajectory returns periodically closer to the centre of the unit circle. This transient behaviour is also reflected by the oscillatory time dependence of the group parameters, especially φ and μ (figures 5(c) and (e), respectively), which remain finite, though they may acquire relatively large absolute values. For $\chi_0 > 2.2$ the dynamics becomes noncompact again, with the phase space trajectory approaching the unit circle without returns to the internal region of the circle. This is shown in figure 6, where we can also see from figures 6(b) that the approaching of the unit circle is not quite monotonic. Note also characteristic behaviour of the group parameters φ and β similar in shape to that of figure 3.

We note that for a periodically modulated amplitude the transition from compact to noncompact trajectories in the phase space occurs at much larger values of χ_0 than in the case of time-independent amplitude. In this latter case transition between the two types of dynamics takes place for $\chi_0 = \omega - \omega_2/2$, i.e. with $\omega_2/\omega = 1.9347$ the transition should occur for $\chi_0 \approx 0.03\omega$. For the amplitude of the pumping field oscillating with frequency $\omega_1 = 1.1256$ the transition occurs at $\chi_0 \approx 2.2\omega$, i.e. two orders of magnitude larger. This is caused by rapid oscillations of the amplitude, and repeating returns of the system to a compact region, where the absolute value of the amplitude is smaller than $\omega - \omega_2/2$. As a result the transition of the dynamics to the noncompact regime is significantly ‘retarded’, in the sense that much larger values of the parameter χ_0 are needed for the trajectories to approach the unit

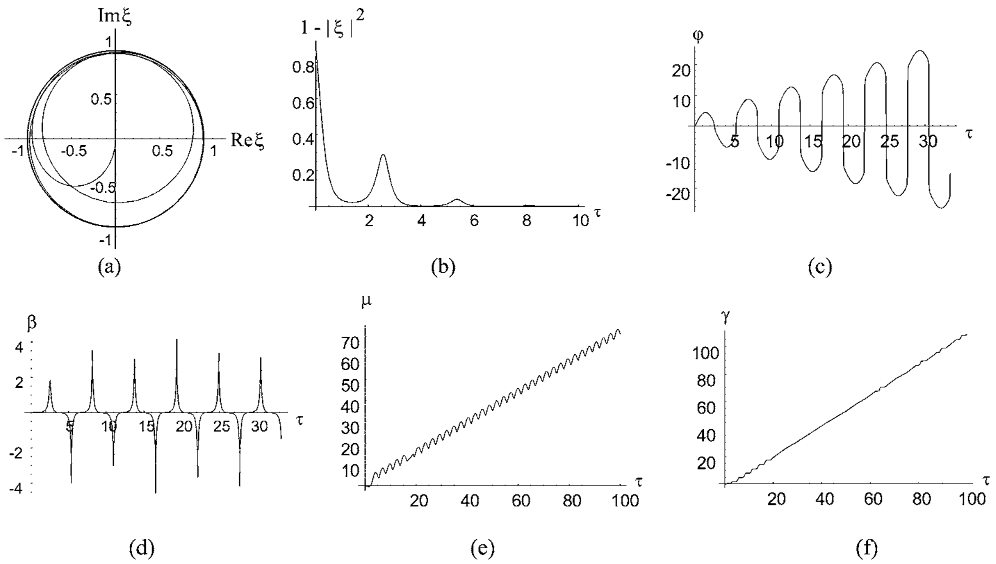

Figure 6. (*a*) Phase space trajectory, (*b*) $1 - |\xi|^2$ and (*c*)–(*f*) time dependence of group parameters φ, β, μ and γ in the noncompact case with $\chi_0 = 2.5$, $\omega_2 = 1.934$ and $\omega_1 = 1.125$. Values of the Floquet coefficients are $\lambda_1 = -7.26$ and $\lambda_2 = -0.14$.

circle. On the other hand, for fixed $\chi_0 > \omega - \omega_2/2$ and varying frequency ω_1 of the amplitude modulation we observe noncompact trajectories for small ω_1, or slowly varying amplitude. With increasing modulating frequency the trajectories become confined to a compact region inside the unit circle, since then χ_0 is not large enough to 'pull' the trajectory to the boundary of the unit circle.

5. Final remarks

We have investigated classical dynamics of the $SU(1,1)$ coherent states using transformations diagonalizing the time-dependent Hamiltonian given as a linear combination of the generators of the $SU(1,1)$ group. The diagonalizing transformation could be chosen to reduce the Hamiltonian either to the compact generator K_0, or to the noncompact generator K_1. An exception is provided by the case of resonant coupling, for which the phase $\alpha = 2\omega t$, and the diagonalized Hamiltonian can be expressed only by K_1, as can be seen from (2.22). Trajectories of the coherent states in the $SU(1,1)$ phase space (Lobachevskii plane) can be divided into two classes: compact and noncompact ones. In the first case the trajectory occupies a compact region inside the unit circle, with diameter smaller than unity. In contrast to this, noncompact trajectories approach the boundary of the phase space with increasing time. In terms of the group transformations diagonalizing the Hamiltonian the character of the phase space trajectory is reflected in the time behaviour of the group transformation parameters. In the compact case the group parameters exhibit oscillatory behaviour with nonincreasing amplitudes, whereas for noncompact trajectories absolute values of the group parameters increase to infinity with increasing time. The parameters of the transformation reducing the Hamiltonian to the compact generator K_0 exhibit an interesting singular behaviour for noncompact trajectories. The parameter φ (cf (2.23)) jumps rapidly from large positive to negative values with large modulus. The parameter β shows characteristic sharp peaks separated by periods of slow variation with small values of the parameters. On the other hand, parameters of the transformation reducing the Hamiltonian to K_1 show much more regular behaviour, with the parameter μ (cf (2.28)) either oscillating (for compact trajectories) or increasing to infinity, and γ always increasing.

Appendix A

We give here transformation formulae for the generators of $SU(1,1)$:

$$e^{i\beta K_2} K_0 e^{-i\beta K_2} = K_0 \cosh\beta - K_1 \sinh\beta \tag{A.1}$$

$$e^{i\beta K_2} K_1 e^{-i\beta K_2} = -K_0 \sinh\beta + K_1 \cosh\beta \tag{A.2}$$

$$e^{i\varphi K_1} K_0 e^{-i\varphi K_1} = K_0 \cosh\varphi + K_2 \sinh\varphi \tag{A.3}$$

$$e^{i\varphi K_1} K_2 e^{-i\varphi K_1} = K_0 \sinh\varphi + K_2 \cosh\varphi \tag{A.4}$$

$$e^{i\gamma K_0} K_1 e^{-i\gamma K_0} = K_1 \cos\gamma - K_2 \sin\gamma \tag{A.5}$$

$$e^{i\gamma K_0} K_2 e^{-i\gamma K_0} = K_1 \sin\gamma + K_2 \cos\gamma. \tag{A.6}$$

Appendix B

We show here a relation between parameters of the group transformation U given by (2.28) and the parameters of the coherent state. The $SU(1,1)$ coherent state can be parametrized as (cf section 2)

$$|\xi\rangle = |\theta, \phi\rangle \tag{B.1}$$

where

$$\xi = -\tanh(\theta/2)e^{-i\phi}. \tag{B.2}$$

To find the time evolution equations fulfilled by θ and ϕ we substitute (B.2) into (2.15) and after separating the real and imaginary parts obtain

$$\dot{\theta} = -2c(t)\sin(\phi - \alpha) \tag{B.3}$$

$$\dot{\phi} = 2\omega - 2c(t)\cos(\phi - \alpha)\coth\theta. \tag{B.4}$$

Consider now the unitary transformation U (2.28) using equation (2.29) to reduce the Hamiltonian to a form containing the group generator K_1. For the parameters μ and γ we obtain

$$\dot{\mu} = -2c(t)\sin\gamma \tag{B.5a}$$

$$\dot{\gamma} = 2\left(\omega - \frac{\dot{\alpha}}{2}\right) - 2c(t)\cos\gamma\coth\mu \tag{B.5b}$$

so that $(\theta, \phi - \alpha)$ and (μ, γ) fulfill the same system of two coupled differential equations. It can be further easily shown that the transformation U with parameters obeying (B.3) and (B.4) leads in fact to vanishing transformed Hamiltonian, so that the transformed state, $|\zeta\rangle$, is time independent and the parameter ζ preserves its initial value ζ_0. Taking for simplicity $\zeta_0 = 0$ and using (3.24) we obtain

$$\xi = -\tanh(\mu/2)e^{-i(\alpha+\gamma)} \tag{B.6}$$

which is the same as (B.2) with $\theta = \mu$ and $\phi = \alpha + \gamma$.

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